

STRONG RATIO LIMIT THEOREMS ASSOCIATED WITH RANDOM WALKS ON UNIMODULAR GROUPS

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ABSTRACT. Strong ratio limit theorems associated with a broad class of spread out random walks on unimodular groups were proved in the preceding paper [8], where these random walks were assumed to have the convergence parameter $R = 1$. In the present paper, we study the case of an arbitrary $R \geq 1$ and clarify the role of the condition that the group is unimodular.

1. INTRODUCTION

The final section of the paper [8] contains strong ratio limit theorems (SRLT) associated with a broad family of random walks on unimodular groups. The conditions imposed in [8] on these random walks in particular guarantee that they are irreducible in the sense of [5] and the convergence parameter R of each of these random walks satisfies $R = 1$. (Following [8], we mainly borrow the probability part of our terminology from [5, 6] and the algebraic part, from [2].) The aim of the present paper is to transfer these results in [8] to the case of arbitrary $R \geq 1$.

In this connection, recall that the SRLT suggested in [8] are slightly different from the traditional ones (e.g., cf. [1, 3–5]). As long as some SRLT of the traditional kind can be applied to a random walk $X = (X_n; n \geq 0)$ on a locally compact group E , this SRLT can be used to obtain information about the existence and values of limits of the form

$$(1) \quad \lim_{n \rightarrow \infty} \frac{\varkappa(P^{n+m}f)}{\mu(P^n g)}, \quad m \geq 0,$$

where P is the transition operator of the random walk and the Borel functions $f, g \geq 0$ defined on E , together with the probability measures \varkappa and μ defined on the set of all Borel subsets of E , satisfy appropriate conditions [1, 3–5]. (Here and in what follows, we write, say, $\varkappa(f) = \int f d\varkappa$ if the integral exists.) However, many authors only consider the case in which these measures are concentrated at some point $x \in E$ and hence the ratio occurring in (1) coincides with $P^{n+m}f(x)/P^n g(x)$ (e.g., see [1]).

Of the conditions included in the traditional SRLT, those implying the existence of limits as $n \rightarrow \infty$ of ratios of the form $\varkappa(P^{n+1}f)/\varkappa(P^n f)$ for one or maybe all admissible functions f play a special role (see [1, 3–5]). The verification of such conditions (apart from the case of symmetric random walks [7]) often encounters serious difficulties. To avoid such conditions as much as possible, we considered SRLT of a more general type in [8]. In these theorems, limits of the form (1)

Date: November 10, 2015.

2010 Mathematics Subject Classification. 60B15.

Key words and phrases. unimodular group, random walk on a group, strong ratio limit theorem.

are replaced by some analogs in which the integer parameter n tends to infinity avoiding some set \mathcal{N} of density 0, that is, a set \mathcal{N} of positive integers such that

$$(2) \quad \lim_{n \rightarrow \infty} \left[\frac{1}{n} q_n(\mathcal{N}) \right] = 0,$$

where $q_n(\mathcal{N})$, $n \geq 0$, is the number of elements $m \in \mathcal{N}$ that do not exceed n . These SRLT were referred to as SRLT with exceptional parameter sets in [8].

The main results of the present paper (see Theorems 1 and 2), which deal with spread out irreducible random walks on unimodular groups, are SRLT of this kind as well. Following [8, 9], we assume that for each of the random walks in question there exists a unique (up to a factor) R -invariant measure taking finite values on compact sets, where R has the same meaning as above, but now the case of $R \neq 1$ is of main interest. The proofs of Theorems 1 and 2 use the results in [8, 9]. As to the assumption that the groups considered are unimodular, it was also introduced in [8], but in the present paper we establish that it is necessary in our theory (see Theorem 3).

In conclusion of the introduction, let us give some notation and definitions. In what follows, E is a locally compact group with countable base, \mathcal{E} is the Borel σ -algebra on E , and π is the right Haar measure on \mathcal{E} ; we also set $\mathcal{E}_+ = \{A \in \mathcal{E} : \pi(A) > 0\}$. The group operation on E is interpreted as multiplication.

On E , we specify a random walk $X = (X_n; n \geq 0)$ with a law v , where v is a probability measure on \mathcal{E} , and a transition operator P . (Thus, X is a homogeneous Markov chain on the measurable space (E, \mathcal{E}) with transition probabilities $p(x, A) = v(x^{-1}A)$, $x \in E$, $A \in \mathcal{E}$.) If, for some positive integer n , the convolution power v^n is nonsingular with respect to π , then X is called a *spread out random walk*. As is shown by [9, Proposition 1], such a random walk is irreducible in the traditional sense [10] if and only if it is irreducible with respect to the measure π in the sense of the theory of irreducible Markov chains [5]. In what follows, unless specified otherwise, the irreducibility of random walks is understood in the traditional sense.

2. AUXILIARY ASSERTIONS

The main aim of this section is to prove the following assertion, which, in particular, contains a full description of small functions and measures [5] for a broad class of random walks.

Proposition 1. *Let a random walk X be spread out, irreducible, and aperiodic in the sense of [5]. Let S_0 be the family of all bounded Borel functions $f \rightarrow [0, \infty)$ such that the set $\{x \in E : f(x) > 0\}$ belongs to \mathcal{E}_+ and is relatively compact. Then*

- (a) *Each function $f \in S_0$ and each measure of the form $\mu = f\pi$ with $f \in S_0$ are small for X . (By definition, $f\pi(A) = \int_A f d\pi$, $A \in \mathcal{E}$.)*
- (b) *If the law v of X is compactly supported, then, conversely, all functions and measures small for X have the form described in (a).*

Proof. By [9, Proposition 3], there exists an open set $U \subset E$ such that the set $sU = \{y \in E : y = sx, x \in U\}$ and the measure $1_{sU}\pi$ are small for X . (Here the symbol 1_A stands for the indicator function of a set $A \in \mathcal{E}$.) Given $f \in S_0$, take a compact set $F \subset E$ such that $f = 0$ outside F . Since E is locally compact, it

follows that this compact set can be covered by finitely many sets of the form sU , $s \in F$. In view of the aperiodicity of X and [5, Corollary 2.1], we can conclude that f is a small function, thus proving the part of (a) pertaining to small functions. A similar argument gives the remaining part of (a).

Now let us proceed to (b). Fix a small function f . If $g \in S_0$, then $aP^m g \geq f$ for some $m \geq 1$ and $a > 0$. Since the measure v^m is now compactly supported, it follows that $P^m g = 0$ outside some compact set and hence $f = 0$ outside the same set. This justifies the part of (b) pertaining to small functions. The case of small measures can be analyzed in a similar way. \square

Corollary 1. *If the conditions indicated in the first sentence of Proposition 1 are satisfied, then every compact set contained in \mathcal{E}_+ is a small set for X .*

Now recall that a measure μ on \mathcal{E} is said (see [9] and many other papers) to be r -invariant for X , $0 < r < \infty$, if $\mu(E) > 0$ and $\mu = r\mu P$, where the measure μP is defined by the formula

$$\mu P(A) = \int \rho(x, A) \mu(dx), \quad A \in \mathcal{E}.$$

Corollary 2. *Under the same assumptions as in Corollary 1, every r -invariant measure μ , $0 < r < \infty$, taking finite values on compact sets is also r -invariant in the sense of [5] (that is, $\mu(A) < \infty$ for at least one and hence for all small $A \subset E$), and moreover, $\mu(A) > 0$ for the same A and for all open $A \subset E$.*

Proof. Every compact set $A_1 \in \mathcal{E}$ is a small set by Corollary 1 and $\mu(A_1) < \infty$ by the assumptions of the corollary; it follows that $0 < \mu(A) < \infty$ for any small A [5, Proposition 5.6]. If a set $B \subset E$ is open, then $\pi(B) > 0$; hence B contains some compact set $A_1 \in \mathcal{E}_+$, and we conclude that $\mu(B) > 0$. \square

3. STRONG RATIO LIMIT THEOREMS

Consider the set $C_0(E)$ of all continuous compactly supported functions on E . We introduce the following condition.

Condition A. For every nonnegative function $f \in C_0(E)$, there exists a $\gamma > 0$ and a $j \geq 1$ such that $v^j \geq \gamma(f\pi)$.

This condition coincides with condition (13A) in [8] and implies, by [8, Lemma 13.1], that the random walk X is spread out, irreducible, and aperiodic. Condition A is necessarily satisfied if X is spread out and irreducible and has a law v whose support is not contained in any coset of any proper normal subgroup of E [6, Chap. 3].

We also need the following condition.

Condition B. There exists a unique (up to a positive factor) R -invariant measure for X taking finite values on compact sets, where R is again the convergence parameter of X .

By using [5, Theorem 5.8 and Proposition 5.6] and Corollary 1, we see that Condition B applies, say, if the random walk in question is spread out, irreducible, and R -recurrent. (The last condition means that X is an R -recurrent Markov chain [5].) It is worth noting that, as [5, Proposition 5.6] and our Corollary 2

show, the measure ν is necessarily absolutely continuous with respect to π under Conditions A and B.

Next, recall that a Borel function $\varphi: E \rightarrow (0, \infty)$ is called an *exponential* on E if $\varphi(xy) = \varphi(x)\varphi(y)$ for any $x, y \in E$ and that Condition B guarantees by [9, Theorem 1] that there exists a unique continuous exponential φ on E such that

$$(3) \quad \int \varphi dv = \frac{1}{R}.$$

In what follows, the symbol φ is used to denote this exponential.

Theorem 1. *Let the group E be unimodular, and let the random walk X satisfy Conditions A and B and have compactly supported law v . Then there exists a subsequence \mathcal{N} of positive integers of density 0 (see (2)) such that*

$$(4) \quad \lim_{\substack{n \rightarrow \infty \\ n \notin \mathcal{N}}} \frac{P^n f(x)}{P^n g(y)} = \frac{\pi(\varphi^{-1}f)\varphi(x)}{\pi(\varphi^{-1}g)\varphi(y)} = \frac{\nu(f)\varphi(x)}{\nu(g)\varphi(y)}$$

for any $x, y \in E$ and $f, g \in C_0(E)$ except for $g \in C_0(E)$ with $\nu(g) = 0$, where ν is the measure indicated in Condition B. Moreover,

$$(5) \quad \lim_{\substack{n \rightarrow \infty \\ n \notin \mathcal{N}}} \frac{P^{n+m}g(y)}{P^n g(y)} = \frac{1}{R^m}, \quad y \in E,$$

for any positive integer m , and the convergence to the limits in (4) and (5) is uniform with respect to x and y ranging over an arbitrary compact set in E .

Proof. Without loss of generality, we assume in what follows that $\nu = \varphi^{-1}\pi$ [9, Theorem 1]. Consider the spread out random walk \tilde{X} on E with the law $\tilde{v} = R\varphi v$. (The relation $\tilde{v}(E) = 1$ follows from (3).) Just as in [9, Sec. 4], we find that the transition operator \tilde{P} of this random walk is related to P by

$$(6) \quad \tilde{P}^n f = R\varphi^{-1}P^n(f\varphi), \quad n \geq 1,$$

where f is an arbitrary bounded Borel function on E . Thus, just as in [9], the operator \tilde{P} is obtained from P by a well-known similarity transformation [5], whence the random walk proves to be irreducible and have the convergence parameter $\tilde{R} = 1$.

Just as for an arbitrary random walk on E , the measure π is invariant for \tilde{X} . Moreover, it is the unique (up to a factor) measure that is invariant for \tilde{X} , is absolutely continuous with respect to π , and takes finite values on compact sets. Indeed, assume that some measure ν_1 has the last three properties but does not coincide with any of the measures $\alpha\pi$, $\alpha > 0$. The one can readily establish the invariance of the measure $\varphi^{-1}\nu_1$ for X with the use of (3). However, this measure differs from any of the measures $\alpha\varphi^{-1}\pi = \alpha\nu$, $\alpha > 0$, and by Condition B this is only possible if $\nu_1(A) = \infty$ for some compact set $A \subset E$. Thus, we arrive at a contradiction with the choice of ν_1 , and hence we have established the desired uniqueness of the measure π .

Next, by Corollary 2, every measure invariant for \tilde{X} is dense in E , that is, takes nonzero values on open sets in E . This, together with the preceding discussion,

permits us to use Theorem 13.1 in [8] and find a subsequence \mathcal{N} of density 0 of positive integers such that

$$(7) \quad \lim_{\substack{n \rightarrow \infty \\ n \notin \mathcal{N}}} \frac{\tilde{P}^n f(x)}{\tilde{P}^n g(y)} = \frac{\pi(f)}{\pi(g)}$$

provided that the functions f and g are nonnegative (which is from now on assumed until we drop this condition) and chosen in accordance with the theorem to be proved (which is assumed until the end of the proof). As to the character of the convergence to the limit in (7), it is as indicated in the theorem.

Consider the special case of $f = \tilde{P}^m g$, $m \geq 1$, where the continuity of $\tilde{P}^m g$ is guaranteed by that of g and the Feller property of random walks [6, Exercise 5.14], while the compactness of its support follows from that of the support of the measure v^m . Relation (7) with $x = y$ can be written as

$$(8) \quad \lim_{\substack{n \rightarrow \infty \\ n \notin \mathcal{N}}} \frac{\tilde{P}^{n+m} g(y)}{\tilde{P}^n g(y)} = 1,$$

because the measure π is invariant for \tilde{X} .

Now we return to (7) and use (6) to derive the desired relations (4) with φf and φg instead of f and g , respectively, from (8). Thus, to obtain (4) in the desired form, we should apply the version of (4) just obtained to $\varphi^{-1}f$ and $\varphi^{-1}g$, respectively, instead of f and g . In a similar way, we derive (5) from (8).

It remains to get rid of the requirement that f and g are nonnegative. Assuming that this requirement is violated, we represent f and g as $f = f_1 - f_2$ and $g = g_1 - g_2$, where $f_1(x) = \max\{0, f(x)\}$, $f_2(x) = -\min\{0, f(x)\}$, and g_1 and g_2 are defined in a similar way. If $\nu(g_1) > 0$, then, by using what was just proved, we successively compute the limits of the ratios

$$\frac{P^n f_i(x)}{P^n g_1(y)}, \quad i = 1, 2, \quad \frac{P^n f(x)}{P^n g_1(y)}, \quad \frac{P^n g_1(x)}{P^n g(y)}$$

as $n \rightarrow \infty$ ($n \notin \mathcal{N}$) and then the limit indicated in (4), after which relations (4) become obvious. The argument for (4) with $\nu(g_2) > 0$ as well as for (5) is equally simple. The proof of the theorem is complete. \square

Remark 1. In the framework of traditional SRLT, one can suggest a natural analog of relations (4) (see Eq. (10) below). Indeed, under appropriate assumptions including those imposed in Theorem 1, the paper [1] permits one to apply the relation

$$(9) \quad \lim_{n \rightarrow \infty} \frac{P^n f(x)}{P^n g(x)} = \frac{\nu(f)}{\nu(g)}$$

for $x \in E$. (We retain the same notation as in Theorem 1.) If we take some $x, y \in E$ and set $z = yx^{-1}$ and $g_z(u) = g(zu)$, $u \in E$, then simple calculations give $P^n g_z(x) = P^n g(y)$ and $\nu(g_z) = \varphi(z)\nu(g) = \varphi(y)\varphi(x^{-1})\nu(g)$. Hence Eq. (9) with g replaced by g_z shows that

$$(10) \quad \lim_{n \rightarrow \infty} \frac{P^n f(x)}{P^n g(y)} = \frac{\varphi(x)\nu(f)}{\varphi(y)\nu(g)}.$$

Thus, we have obtained the desired analog of Eqs. (4).

The following assertion again uses the measure ν occurring in Condition B, but the sequence \mathcal{N} may differ from the one considered in Theorem 1.

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. Then there exists a sequence \mathcal{N} of positive integers of density 0 such that*

$$(11) \quad \lim_{\substack{n \rightarrow \infty \\ n \notin \mathcal{N}}} \frac{\varkappa(P^n f)}{\mu(P^n g)} = \frac{\varkappa(\varphi)\pi(\varphi^{-1}f)}{\mu(\varphi)\pi(\varphi^{-1}g)} = \frac{\varkappa(\varphi)\nu(f)}{\mu(\varphi)\nu(g)},$$

$$(12) \quad \lim_{\substack{n \rightarrow \infty \\ n \notin \mathcal{N}}} \frac{\varkappa(P^{n+m}g)}{\mu(P^n g)} = \frac{1}{R^m}, \quad m \geq 1,$$

where f and g are bounded Borel functions defined on E and vanishing outside some compact set, $\nu(g) \neq 0$, and the probability measures μ and \varkappa defined on \mathcal{E} have the form indicated in Proposition 1 (a).

Proof. Assume for now that the functions occurring in the theorem are continuous. By applying (4) and the standard rule of passing to the limit in the integrand, we find that

$$\frac{\varkappa(P^n f)}{P^n g} \longrightarrow \frac{\nu(f)\varkappa(\varphi)}{\nu(g)\varphi}$$

everywhere in E as $n \rightarrow \infty$ ($n \notin \mathcal{N}$), where \mathcal{N} is so far chosen in the same way as in Theorem 1. From this, we derive (11) by a similar argument provided that $\nu(f) \neq 0$. If, on the other hand, $\nu(f) = 0$, then, in view of what has already been proved, we can use (11) with the functions $f_m = f + \frac{1}{m}g$, $m \geq 1$, instead of f and then pass to the limits as $m \rightarrow \infty$ to obtain (11) again with the original function f . Thus, (11) applies to any functions f and g of the form we have just spoken of.

Now we fix Borel functions $f, g \geq 0$ and measures μ and \varkappa satisfying the assumptions of the theorem and find continuous functions $f_n, g_n \geq 0$ vanishing outside a common compact set and satisfying $\pi(|f - f_n|) \rightarrow 0$ and $\pi(|g - g_n|) \rightarrow 0$ as $n \rightarrow \infty$. It follows from the first relation in [8, Eq. (12.8)] that

$$\liminf \frac{\varkappa(P^n f)}{\mu(P^n g)} \geq \frac{1 - \varepsilon}{1 + \varepsilon} \liminf \frac{\varkappa(P^n f_m)}{\mu(P^n g_m)}$$

for any given $\varepsilon \in (0, 1)$ and all sufficiently large $m \geq 1$. (Here and in the following, we assume that a subsequence \mathcal{N} of positive integers of density 0 is chosen in an appropriate way [8] and that the lower limits correspond to $n \rightarrow \infty$, $n \notin \mathcal{N}$.) According to what was said at the beginning of the proof, the second of these lower limits is equal to the ratio

$$\frac{\varkappa(\varphi)\nu(f_m)}{\mu(\varphi)\nu(g_m)},$$

so that if we first let $m \rightarrow \infty$ and then let $\varepsilon \rightarrow 0$, then we conclude from the last inequality that

$$(13) \quad \liminf \frac{\varkappa(P^n f)}{\mu(P^n g)} \geq \frac{\varkappa(\varphi)\nu(f)}{\mu(\varphi)\nu(g)}.$$

Just in the same way, for $\nu(f) > 0$ the similar lower limit of the ratio $\mu(P^n f)/\varkappa(P^n f)$ cannot be less than $1/\delta$, where δ is equal to the right-hand side of (13). From this and (13), we readily derive (11) (cf. the proof of Theorems 12.4 and 13.2 in [8]).

If $f = 0$ ν -a.e., then, to achieve the same goal, it suffices to compare the results of the substitution of the functions $f + g$ and g for f into (11).

In the special case of $f = P^m g$, (11) implies (12) (cf. the proof of (8); here it is recommended to take into account the R -invariance of the measure ν).

Finally, just as in the preceding proof, one can readily drop the requirement that f and g be nonnegative, thus completing the proof of Theorem 2. \square

Remark 2. If $f, g \in C_0(E)$ and $\nu(g) = 0$, then one can apply relations (11) and (12) with arbitrary compactly supported probability measures μ and \varkappa defined on \mathcal{E} . To prove this, it suffices to note that the first paragraph of the proof of Theorem 2 does not use the possibility of representation of these measures in the form indicated in the theorem.

In conclusion, it remains to explain why one cannot omit the assumption that the group E is unimodular from Theorems 1 and 2.

Theorem 3. *If the group E is not unimodular, then there does not exist an irreducible spread out random walk on E with any convergence parameter $R \geq 1$ possessing a unique (up to a factor) R -invariant measure taking finite values on compact sets.*

Proof. We restrict ourselves to the case of $R = 1$, because otherwise one could consider the random walk \tilde{X} with convergence parameter $\tilde{R} = 1$ instead of X (see the proof of Theorem 1).

Let X be a random walk on X with the corresponding properties possessing a unique (up to a factor) invariant measure taking finite values on compact sets. Clearly, this measure coincides with π up to a factor. Next, assume that functions $f, g \geq 0$ belong to the family $C_0(E)$ and $\pi(g) \neq 0$. By [8, Eq. (13.13)],

$$(14) \quad \lim_{n \in \mathcal{N}} \frac{P^n f(e)}{P^n g(e)} = \frac{\pi(f)}{\pi(g)},$$

where \mathcal{N} is an appropriate subsequence of positive integers of density 0, e is the identity element of the group, and the limit in (14)–(16) is taken as $n \rightarrow \infty$, $n \notin \mathcal{N}$; we point out that, in contrast to the text adjacent to the proof of [8, Eq. (13.13)], the derivation of [8, Eq. (13.13)] itself does not use the assumption that E is unimodular. Hence, in view of [2, Sec. 15.27(c)], we obtain

$$(15) \quad \lim_{n \in \mathcal{N}} \frac{P^n g(x)}{P^n g(e)} = \lim_{n \in \mathcal{N}} \frac{P^n g_x(e)}{P^n g(e)} = \frac{\pi(g_x)}{\pi(g)} = \Delta(x)$$

for $x \in E$, where Δ is the modular function of E and the translate g_x of g is defined by analogy with g_z (see Remark 1).

Now take a compact set $F \subset E$. By [8, Lemmas 13.3 and 13.4], the family of functions $\{P^n g/P^n g(e); n \geq n_1\}$ with sufficiently large positive integer n_1 is equicontinuous on F . Consequently, the convergence in (15) is uniform with respect to $x \in F$. Hence, by taking the support of v for F , we obtain, in view of (15),

$$(16) \quad \begin{aligned} \lim_{n \in \mathcal{N}} \frac{P^{n+1} g(x)}{P^n g(x)} &= \lim_{n \in \mathcal{N}} \int \frac{P^n g(xy)}{P^n g(x)} v(dy) \\ &= \int \Delta(xy) v(dy) = \Delta(x) \int \Delta dv = r^{-1} \Delta(x), \end{aligned}$$

because Δ is a continuous exponential on E [2, Theorem 15.11]; here $r = [\int \Delta dv]^{-1}$.

By [8, Lemma 13.2], the first of the limits in (16) cannot be less than 1, and hence (16) implies the inequality $\Delta(x) \geq r$, $x \in E$. At the same time, we have $\Delta(x^{-1}) = \Delta^{-1}(x) \geq r$, so that $r \leq \Delta(x) \leq r^{-1}$, $x \in E$, whence it follows that $\Delta(x) \equiv 1$. (Indeed, if $\Delta(x) \neq 1$ for some $x \in E$, then, for sufficiently large positive integer n , either $\Delta(x^n) = [\Delta(x)]^n < r$ or $\Delta(x^n) > r^{-1}$, but both of the last inequalities are excluded by our argument.) Thus, we have established that the group E is unimodular (cf. [2, Remark 15.12]). \square

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